Goal: Determine correlation for optimum economic insulation thickness on a flat plate?

Given: Annual fixed charges per square meter are directly proportional to thickness.
(a) neglecting air film resistance = assume constant surface temperature
(b) including convection air film. The air-film heat transfer coefficient is constant for all thicknesses of insulation.

Approach: diagram the system to clarify situations.
Define parameter nomenclature and units for each parameter.
Derive annual cost equation as function of thickness which will include terms for annualized fixed costs and heat loss operating costs.
Differentiate annual cost equation, set equal to zero, algebraically rearrange into solution form.

Diagram: T_A ambient air.

$Q = h_{c\text{-air}} A \, \Delta(T_A - T_2)$

T_2 top surface temperature

$Q = k A \, \Delta(T_2 - T_S) / x$

Flat Plate surface T_S constant uniform temperature.

Parameters: (SI units)

$\Delta T\ °K = T_A - T_S$ Temperature driving force { alt. English units {
x insulation thickness (meters) { ft }
k insulation thermal conductivity (W / m - °K) { Btu / hr - ft - °F }
q heat loss per unit area (Watts / m^2) { Btu / hr - ft^2 }
H hours of operation per year (hr / yr) { hr / yr }
C_H cost of heat ($ / J) { $ / Btu }
C_{FC} annual fixed charges per thickness ($ / yr - m^2 - m) { $ / yr - ft^2 - ft }
A surface area (m^2) { ft^2 }

$$Q = k A \Delta T / x$$ heat loss through insulation

$h_{c\text{-air}}$ air film convection heat transfer coefficient (W / m^2 - °K) for part b
U overall heat transfer coefficient (W / m^2 - °K) { Btu / hr – ft^2 - °F }
where $1/U = x/k + 1/h_{c\text{-air}}$

$$Q = U A \Delta T$$ heat loss from plate to ambient air

Basis: per unit area, divide by A to put everything on per m^2 basis

Investment Cost: ≈ directly proportional to thickness and straight line depreciation
$C_{installed}$ = ($ / m^2 - m); capital recovery depreciation factor, e = 1 / yr
$C_{FC} = e\, C_{installed}$ = ($ / yr - m^2 - m)

∴ Annual fixed charges: = $x C_{FC}$ (m) ($ / yr - m^2 - m) [$ / yr - m^2]

(a) Derivation: neglect air film resistance = assume constant air side insulation surface Temperature = T_A ambient air

Determine optimum thickness to minimize annual cost, at steady state

Cost of heat loss: $q = Q/A = kA\Delta T/x/A = k\Delta T/x$

$\therefore$ Annual operating charges: $= q\,H\,C_H$ (Watts/m^2) (hr/yr) (\$/J) [\$/yr-m^2]

$\therefore$ Total Annual Costs: $T_C = x\,C_{FC} + H\,C_H\,k\,\Delta T/x$ [\$/yr-m^2]

Differentiate with respect to x and set equal to zero:

$$\therefore \frac{dT_C}{dx} = C_{FC} - H\,C_H\,k\,\Delta T/x^2 = 0$$

Solve for optimum thickness, x:

$$\therefore x^2 = k\,\Delta T\,H\,C_H/C_{FC}$$

$$\therefore x_{optimum} = [k\,\Delta T\,H\,C_H/C_{FC}]^{1/2}$$

$\Leftrightarrow$ Answer part (a)

(b) Derivation: include air film resistance = assume convection coefficient

where $1/U = x/k + 1/h_{c\text{-}air}$

$$\therefore U = k\,h_{c\text{-}air}/(k + x\,h_{c\text{-}air})$$

Cost of heat loss: $\approx q = k\,h_{c\text{-}air}\,\Delta T/(k + x\,h_{c\text{-}air})$

$\therefore$ Total Annual Costs: $T_C = x\,C_{FC} + H\,C_H\,k\,h_{c\text{-}air}\,\Delta T/(k + x\,h_{c\text{-}air})$

Differentiate with respect to x and set equal to zero:

$$\therefore \frac{dT_C}{dx} = C_{FC} - h_{c\text{-}air}\,H\,C_H\,k\,h_{c\text{-}air}\,\Delta T/(k + x\,h_{c\text{-}air})^2 = 0$$

Solve for optimum thickness, x:

$$\therefore (k + x\,h_{c\text{-}air})^2 = h^2_{c\text{-}air}\,k\,\Delta T\,H\,C_H/C_{FC}$$

$$(x\,h_{c\text{-}air})^2 = h^2_{c\text{-}air}\,k\,\Delta T\,H\,C_H/C_{FC} - k^2$$

$$\therefore x_{optimum} = [k\,\Delta T\,H\,C_H/C_{FC}]^{1/2} - k/h_{c\text{-}air}$$

$\Leftrightarrow$ Answer part (b)